

Causality of singular linear discrete time systems

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Abstract: We consider two type of systems, a linear singular discrete time system and a linear singular fractional discrete time system whose coefficients are square constant matrices. By assuming that the input vector changes only at equally space sampling instants we investigate and provide properties for causality between state and inputs and causality between output and inputs.

Keywords : causality, linear, fractional, singular, system.

1 Introduction

Many authors have studied generalized discrete & continuous time systems, see [1-28], and their applications, see [29-35]. Many of these results have already been extended to systems of differential & difference equations with fractional operators, see [36-45].

If we define \mathbb{N}_α by $\mathbb{N}_\alpha = \{\alpha, \alpha + 1, \alpha + 2, \dots\}$, α integer, and n such that $0 < n < 1$ or $1 < n < 2$, then the nabla fractional operator in the case of Riemann-Liouville fractional difference of n -th order for any $Y_k : \mathbb{N}_a \rightarrow \mathbb{R}^m$ is defined by

$$\nabla_\alpha^{-n} Y_k = \frac{1}{\Gamma(n)} \sum_{j=\alpha}^k (k-j+1)^{\overline{n-1}} Y_j.$$

We denote $\mathbb{R}^{m \times 1}$ with \mathbb{R}^m . Where the raising power function is defined by

$$k^{\bar{\alpha}} = \frac{\Gamma(k+\alpha)}{\Gamma(k)}.$$

We consider the singular discrete time system of the form

$$FY_{k+1} = GY_k + V_k, \quad k = 1, 2, \dots, \quad (1)$$

the singular fractional discrete time system of the form

$$F\nabla_0^n Y_k = GY_k + V_k, \quad k = 1, 2, \dots, \quad (2)$$

the output system

$$X_k = CY_k, \quad k = 1, 2, \dots, \quad (3)$$

and the known initial conditions (IC)

$$Y_0. \quad (4)$$

Where $F, G \in \mathbb{R}^{r \times m}$, $Y_k \in \mathbb{R}^m$, $V_k \in \mathbb{R}^r$, $X_k \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n \times r}$. The matrices F , G can be non-square ($r \neq m$) or square ($r = m$) with F singular ($\det F = 0$).

2 Preliminaries

Throughout the paper we will use in several parts matrix pencil theory to establish our results. A matrix pencil is a family of matrices $sF - G$, parametrized by a complex number s , see [46-53].

Definition 2.1. Given $F, G \in \mathbb{R}^{r \times m}$ and an arbitrary $s \in \mathbb{C}$, the matrix pencil $sF - G$ is called:

1. Regular when $r = m$ and $\det(sF - G) \neq 0$;
2. Singular when $r \neq m$ or $r = m$ and $\det(sF - G) \equiv 0$.

In this article we consider the system (1) with a *regular pencil*, where the class of $sF - G$ is characterized by a uniquely defined element, known as the Weierstrass canonical form, see [46-53], specified by the complete set of invariants of $sF - G$. This is the set of elementary divisors of type $(s - a_j)^{p_j}$, called *finite elementary divisors*, where a_j is a finite eigenvalue of algebraic multiplicity p_j ($1 \leq j \leq \nu$), and the set of elementary divisors of type $\hat{s}^q = \frac{1}{s^q}$, called *infinite elementary divisors*, where q is the algebraic multiplicity of the infinite eigenvalue. $\sum_{j=1}^{\nu} p_j = p$ and $p + q = m$.

From the regularity of $sF - G$, there exist non-singular matrices $P, Q \in \mathbb{R}^{m \times m}$ such that

$$PFQ = \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix}, \quad (5)$$

$$PGQ = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}.$$

J_p, H_q are appropriate matrices with H_q a nilpotent matrix with index q , J_p a Jordan matrix and $p + q = m$. With $0_{q,p}$ we denote the zero matrix of $q \times p$. The matrix Q can be written as

$$Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \quad (6)$$

$Q_p \in \mathbb{R}^{m \times p}$ and $Q_q \in \mathbb{R}^{m \times q}$. The matrix P can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}. \quad (7)$$

$P_1 \in \mathbb{R}^{p \times r}$ and $P_2 \in \mathbb{R}^{q \times r}$. The following results have been proved.

Theorem 2.1. (See [1-28]) We consider the systems (1) with a regular pencil. Then, its solution exists and for $k \geq 0$, is given by the formula

$$Y_k = Q_p J_p^k D + Q D_k.$$

Where $D_k = \begin{bmatrix} \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i \\ - \sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i} \end{bmatrix}$ and $D \in \mathbb{R}^p$ is a constant vector. The matrices Q_p , Q_q , P_1 , P_2 , J_p , H_q are defined by (5), (6), (7).

Theorem 2.2. (See [36-45]) We consider the systems (2) with a regular pencil. Then, its solution exists if and only if all finite eigenvalues of the pencil are distinct and lie within the open disk $S = \{s \in \mathbb{R} : |s| < 1\}$; Then, the solution of system (2) for $k \geq 0$, is given by the formula

$$Y_k = Q_p (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)D + Q D_k.$$

Where $D_k = \begin{bmatrix} \sum_{i=1}^k (k-i+1)^{\overline{n-1}} F_{n,n}(J_p(k+n-i)^{\bar{n}}) P_1 V_i \\ - \sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_k \end{bmatrix}$ and $D \in \mathbb{R}^p$ is a constant vector. The matrices Q_p , Q_q , P_1 , P_2 , J_p , H_q are defined by (5), (6), (7). The discrete Mittag-Leffler function with two parameters $F_{n,n}(J_p(k+n)^{\bar{n}})$ is defined by $F_{n,n}(J_p(k+n)^{\bar{n}}) = \sum_{i=0}^{\infty} J_p^i \frac{(k+n)^{\overline{in}}}{\Gamma((i+1)n)}$.

Definition 2.2. Consider the system (1) and (2) with known IC of type (4). Then the IC are called consistent if there exists a solution for the system (1) and (2) respectively which satisfies the given conditions.

Proposition 2.1. The IC of systems (1) and (2) are consistent if and only if

$$Y_0 \in \text{colspan} Q_p + Q D_0.$$

Proposition 2.2. Consider the system (1) with given IC. Then the solution for the initial value problem (1), (4) is unique if and only if the IC are consistent. Then, the unique solution is given by the formula

$$Y_k = Q_p J_p^k Z_0^p + Q D_k.$$

where $D_k = \begin{bmatrix} \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i \\ - \sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i} \end{bmatrix}$ and Z_0^p is the unique solution of the algebraic system $Y_0 = Q_p Z_0^p + D_0$.

Proposition 2.3. Consider the system (1) with given IC (4). Then if there exists a solution for the initial value problem, it is unique if and only if the IC are consistent. Then, the unique solution is given by the formula

$$Y_k = Q_p (k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p + Q D_k.$$

Where $D_k = \begin{bmatrix} \sum_{i=1}^k (k-i+1) \overline{n-1} F_{n,n}(J_p(k+n-i)^{\bar{n}}) P_1 V_i \\ - \sum_{i=0}^{q^*-1} \nabla_0^{in} H_q^i P_2 V_k \end{bmatrix}$ and Z_0^p is the unique solution of the algebraic system $Y_0 = Q_p Z_0^p + D_0$.

3 Causality

Generally for systems of type (1), (2) we define the notion of causality, which is properly defined bellow

Definition 3.1. The singular systems (1), (2) are called causal, if its state Y_k , for any $k > 0$ is determined completely by initial state Y_0 and former inputs V_0, V_1, \dots, V_k . Otherwise it is called non-causal.

Next we will present the causality in the singular systems of the form (1), (2).

Proposition 3.1. For system (1) causality between state and inputs exists if and only if there exists a matrix $B \in \mathbb{R}^{r \times r_1}$ such that $V_k = BU_k$ and $H_q P_2 B = 0_{q,r_1}$. Where $U_k \in \mathbb{R}^{r_1}$, while causality between output and input exists if and only if every column of the matrix $\begin{bmatrix} Q_q H_q P_2 B & \dots & Q_q H_q^{q^*-1} P_2 B \end{bmatrix}$ lies in the right nullspace of the matrix C . The fractional singular systems of the form (2) is characterized by the property of causality.

Proof. From Proposition 2.2 the solution of system (1) with IC (4) is given by

$$Y_k = Q_p J_p^k Z_0^p + Q D_k,$$

or, equivalently,

$$Y_k = Q_p J_p^k Z_0^p + Q_p \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i - Q_q \sum_{i=0}^{q^*-1} H_q^i P_2 V_{k+i}.$$

It is clear that causality of (1) depends on the term $\sum_{i=0}^{q^*-1} Q_q H_q^i P_2 V_{k+i}$ and obviously Y_k for any $k \geq 0$ is to be determined by former inputs if and only if there exists a matrix $B \in \mathbb{R}^{r \times r_1}$ such that $V_k = BU_k$ and $Q_q H_q^i P_2 B = 0_{q,r_1}$, or, equivalently, $H_q^i P_2 B = 0_{q,r_1}$, since Q_p has linear independent columns it has a left inverse, or, equivalently, $H_q P_2 B = 0_{q,r_1}$. From (3) setting Y_k in the state output X_k we get

$$X_k = C Q_p J_p^{k-0} Z_0^p + C Q_p \sum_{i=0}^{k-1} J_p^{k-i-1} B_p V_i - C Q_q \sum_{i=0}^{q^*-1} H_q^i P_2 V_{k+i}.$$

From the above expression it is clear that non-causality is due to the existence of the term $\sum_{i=0}^{q^*-1} C Q_q H_q^i P_2 V_{k+i} = \sum_{i=0}^{q^*-1} C Q_q H_q^i P_2 B U_{k+i}$. So the causal relationship between X_k and V_k exists if and only if $C Q_q H_q^i P_2 B = 0_{m,r_1}$ for every $i = 1, 2, \dots, q^* - 1$. This can be written equivalently as $C \begin{bmatrix} Q_q H_q P_2 B & \dots & Q_q H_q^{q^*-1} P_2 B \end{bmatrix} = 0_{m,q^* n r_1}$ and thus system (1) is causal if and only if every column of the matrix $\begin{bmatrix} Q_q H_q P_2 B & \dots & Q_q H_q^{q^*-1} P_2 B \end{bmatrix}$

lies in the right nullspace of the matrix C .

From Proposition 2.3 the solution of system (2) with IC (4) is given by

$$Y_k = Q_p(k+1)^{\overline{n-1}} F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)Z_0^p + QD_k,$$

or, equivalently,

$$Y_k = Q_p J_p^k Z_0^p + Q_p \sum_{i=1}^k (k-i+1)^{\overline{n-1}} F_{n,n}(J_p(k+n-i)^{\bar{n}}) P_1 V_i - Q_q \sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_k.$$

It is clear that Y_k of (2) is determined completely by the IC Y_0 (Z_0^p is the unique solution of the linear system $Y_0 = Q_p Z_0^p + D_0$) and the former inputs V_0, V_1, \dots, V_k and that occurs also between output and input of this system. The proof is completed.

Conclusions

In this article we focused and provided properties for causality of two types of system. A linear singular discrete time system and a linear singular fractional discrete time system whose coefficients are square constant matrices.

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